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CONTACT STRESSES AT THE APEX OF A WEDGE-SHAPED PUNCH PRESSED UPON THE EDGE OF AN ELASTIC THREE-DIMENSIONAL WEDGE†

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An asymptotic method, used previously for a similar problem [1], is used to study the behaviour of contact stresses at a new singular point—the intersection of the apex of a wedge-shaped punch with the edge of an elastic three-dimensional wedge. Friction in the contact region is ignored. At fairly small wedge angles of the punch and relatively large angles of the elastic wedge, the leading term in the expansion of the contact pressures near the singular point $r=0$ is an oscillator $r^{-3/2}\cos(\theta\ln r)$, where θ depends mainly on the wedge angle of the punch. If, however, the angles of the punch and the elastic wedge are of the same order of magnitude, terms $r^{-\omega_1-3/2-i\omega_2}$, $0<\omega_1<1/2$, may appear, which may cause stronger oscillations of the contact pressures near the punch apex if $\omega_2\neq 0$.

1. The action of an absolutely rigid punch of wedge planform, pressed into an elastic half-space, was considered in [1–7]. Those publications were concerned mainly with determining the singularities of the contact pressures at the wedge apex. The most complicated cases, taking into account friction or cohesion between a punch of arbitrary wedge angle and a half-space, were studied in [4]. In problems with mixed boundary conditions it is interesting to determine the singularities of the stresses at corner points of an even more complex type [8].

We consider a punch whose planform is a region Ω, pressed into the edge of an elastic three-dimensional wedge with angle α $(0<\alpha<2\pi)$ on whose other edge one of the following boundary conditions holds (no stress, sliding or rigid bonding, r, φ, z are cylindrical coordinates, and the z axis points along the edge of the wedge)

$$\begin{aligned}&(\text{a})\ \sigma_\varphi=\tau_{r\varphi}=\tau_{\varphi z}=0\\&(\text{b})\ u_\varphi=\tau_{r\varphi}=\tau_{\varphi z}=0\\&(\text{c})\ u_r=u_\varphi=u_z=0\end{aligned}\tag{1.1}$$

This problem reduces to the following integral equation [9, 10]

$$\frac{1-\nu}{G}\frac{2}{\pi^3}\iint_\Omega q(x,y)\,dx\,dy\int_0^\infty\int_0^\infty \operatorname{sh}\pi u W_m(u)K_{iu}(tr)\times$$
$$\times\left[K_{iu}(tx)+\frac{1}{\operatorname{ch}\pi u/2}B_m^u\left\{\operatorname{ch}\frac{\pi y}{2}K_{iy}(tx)\right\}\right]\cos t(z-y)\,du\,dt=f(r,z),\quad (r,z)\in\Omega\tag{1.2}$$

(a) $m=1$, (b) $m=2$, (c) $m=3$

$$W_1(u)=\frac{\operatorname{sh}2\alpha u+u\sin 2\alpha}{\operatorname{ch}2\alpha u-u^2(1-\cos 2\alpha)+1},\quad W_2(u)=\frac{\operatorname{ch}2\alpha u-\cos 2\alpha}{\operatorname{sh}2\alpha u+u\sin 2\alpha}$$

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$$W_3(u)=\frac{\kappa\,\mathrm{sh}\,2\alpha u-u\sin 2\alpha}{\kappa\,\mathrm{ch}\,2\alpha u+u^2(1-\cos 2\alpha)+(1+\kappa^2)/2},\quad \kappa=3-4\nu$$

$$B_1^u=\frac{W_+(u)}{2W_1(u)}B_+^u-\frac{W_-(u)}{2W_1(u)},\quad W_\pm(u)=\pm\frac{\mathrm{ch}\,2\alpha u\mp\cos\alpha}{\mathrm{sh}\,2\alpha u\pm u\sin\alpha}$$

$$B_{\pm,2,3}^u=\sum_{n=1,}^{\infty}(1-2\nu)^n(A_{\pm,2,3}^u)^n$$

$$A_{\pm,2,3}^u\{f(y)\}=\int_0^\infty L_{\pm,2,3}(u,y)f(y)\,\mathrm{d}y$$

$$L_{\pm,2,3}(u,y)=2\,\mathrm{ch}\frac{\pi u}{2}\mathrm{sh}\frac{\pi y}{2}W_{\pm,2,3}(y)\int_0^\infty\frac{\mathrm{sh}\,\pi t g_{\pm,2,3}(t)\,\mathrm{d}t}{(\mathrm{ch}\,\pi t+\mathrm{ch}\,\pi u)(\mathrm{ch}\,\pi t+\mathrm{ch}\,\pi y)}$$

$$g_\pm(t)=\begin{Bmatrix}\mathrm{cth}\,\alpha t/2\\ \mathrm{th}\,\alpha t/2\end{Bmatrix}\frac{\sin^2\alpha}{\mathrm{ch}\,\alpha t\mp\cos 2\alpha},\quad g_2(t)=\frac{\mathrm{cth}\,\alpha t\sin^2 2\alpha}{\mathrm{ch}\,\alpha t\mp\cos 4\alpha}\tag{1.3}$$

$$g_3(t)=-\frac{\mathrm{th}\,\alpha t\sin^2 2\alpha}{\mathrm{ch}\,2\alpha t+\cos 4\alpha}+\sin^2\alpha\{f_1(t)[2f_2(t)-tf_3(t)]-$$
$$-f_4(t)[2f_3(t)+tf_2(t)]\}/f_5(t)-2(1-\nu)\sin\alpha\{f_1(t)(\sin 3\alpha-$$
$$-\sin\theta\,\mathrm{ch}\,2\alpha t)-f_4(t)\cos\alpha\,\mathrm{sh}\,2\alpha t\}/f_5(t)$$

$$f_1(t)=\kappa\,\mathrm{sh}\,2\alpha t\cos 2\alpha-t\sin 2\alpha,\ f_2(t)=\cos 2\alpha+\sin^2 2\alpha-\mathrm{ch}\,2\alpha t$$
$$f_3(t)=\sin 2\alpha\,\mathrm{th}\,\alpha t(1+\cos 2\alpha),\ f_4(t)=\sin 2\alpha(\kappa\,\mathrm{ch}\,2\alpha t-1)$$
$$f_5(t)=[f_1^2(t)+f_4^2(t)](\mathrm{sh}^2\,\alpha t+\cos^2 2\alpha)$$

Here, $f(r, z)$ is a function defining the shape of the punch base and the degree to which it penetrates into the wedge, $q(r,z)$ are the contact stresses under the punch, and G and ν are the shear modulus and Poisson's ratio, respectively, of the wedge material.

As shown in [9], the Neumann series B_m^u $(m=1,\ 2,\ 3)$ in formulae (1.2) and (1.3) are uniformly convergent in the space $C_M(0,\ \infty)$ of bounded continuous functions on the half-line for all practically significant Poisson's ratios $\nu>\nu_*(\alpha)$, i.e. usually for $\nu_*(\alpha)$ values close to zero.

Let Ω be an infinite wedge of angle 2β $(0<\beta<\pi/2)$, described in polar coordinates ρ, ψ, $(r=\rho\cos\psi,\ z=\rho\sin\psi)$ by inequalities $0\leqslant\rho<\infty$, $|\psi|\leqslant\beta$. To eliminate solutions of Eq. (1.2) with infinite energy, we shall confine our attention to the case in which both functions $q_*(\rho,\ \psi)=(1-\nu)q(r,\ z)/G$ and $f_*(\rho,\ \psi)=f(r,\ z)$ have Mellin transforms with variable ρ and

$$\int_{-\beta}^{\beta}\mathrm{d}\psi\int_0^\infty\left|\begin{Bmatrix}q_*(\rho,\psi)\\ f_*(\rho,\psi)\end{Bmatrix}\right|\rho\,\mathrm{d}\rho<\infty\tag{1.4}$$

Now, writing Eq. (1.2) in terms of ρ and ψ and taking Mellin transforms with respect to ρ of both sides, we obtain [11]

$$\int_{-1}^{1}q_s(\xi)K_s\left(\frac{\xi}{\lambda},\frac{x}{\lambda}\right)\mathrm{d}\xi=\pi f_s(x),\quad |x|\leqslant 1\tag{1.5}$$

$$K_s(t,p)=\frac{\pi}{2\cos\pi s}P_{s-\frac{1}{2}}(-\cos(t-p))+\frac{1}{2}\int_0^\infty \mathrm{sh}\pi u(W_m(u)-\mathrm{dh}\,\pi u)\times$$
$$\times[R_+(-s,u,t)R_+(s,u,p)+R_-(-s,u,t)R_-(s,u,p)]\mathrm{d}u+$$
$$+\int_0^\infty\mathrm{sh}\frac{\pi u}{2}W_m(u)\left[R_+(s,u,p)B_m^u\left\{\mathrm{ch}\frac{\pi y}{2}R_+(-s,u,t)\right\}+\right.\tag{1.6}$$

$$+R_-(s,u,p)B_m^u\left\{\operatorname{ch}\frac{\pi y}{2}R_-(-s,u,t)\right\}\Bigg]du \quad (|\operatorname{Re} s|<\tfrac{1}{2})$$

$$R_\pm(s,u,t)=\frac{1}{2}\Gamma\left(\frac{1}{2}+s+iu\right)\begin{Bmatrix}\operatorname{cosec}[\pi(\tfrac{1}{2}+s-iu)/2]\\ \sec[\pi(\tfrac{1}{2}+s-iu)/2]\end{Bmatrix}\times$$

$$\times[P_{s-1/2}^{-iu}(\sin t)\pm P_{s-1/2}^{-iu}(-\sin t)]$$

$$x=\psi/\beta,\quad \lambda=1/\beta, q_s(x)=q_s^*(\psi), f_s(x)=f_s^*(\psi)/\beta$$

$$\frac{1}{2\pi r}\int_\Gamma q_s^*(\psi)\rho^{-s-3/2}ds=q_*(\rho,\psi),\quad \frac{1}{2\pi i}\int_\Gamma f_s^*(\psi)\rho^{-s-1/2}ds=f_*(\rho,\psi)$$

Here Γ is a straight line parallel to the imaginary axis in the complex s-plane, $\Gamma(s)$ is the Gamma function, and $P_s^\mu(x)$ are spherical functions [12].

The derivation of formula (1.6) for $K_s(t, p)$ uses the value of the integral for a half-space ($m=1$, $\alpha=\pi$) [3]

$$\begin{aligned}&\int_0^\infty \operatorname{ch}\pi u[R_+(-s,u,t)R_+(s,u,p)+R_-(-s,u,p)]du=\\&=\frac{\pi}{\cos\pi s}P_{s-1/2}(-\cos(t-p)),\quad |\operatorname{Re} s|<\tfrac{1}{2}\end{aligned}\tag{1.7}$$

Lemma. The kernel (1.6) of the integral equation (1.5) is symmetric, i.e. $K_s(t, p)=K_s(p, t)$.

The proof, which is based on the equality [12]

$$P_{s-1/2}^\mu(x)=P_{-s-1/2}^\mu(x)$$

relies on changing the order of integration and changing the variables of integration in each term of the Neumann series B_m^u occurring in the expression for $K_s(t, p)$.

The kernel (1.6) admits of the following asymptotic expansion

$$K_s\left(\frac{\xi}{\lambda},\frac{x}{\lambda}\right)=\sum_{n=0}^\infty\frac{1}{\lambda^{2n}}\left\{\left(a_n(s)+b_n(s)\ln\left|\frac{\xi-x}{\lambda}\right|\right)(\xi-x)^{2n}+f_{2n}^s(\xi,x)\right\}\tag{1.8}$$

where $f_{2n}^s(\xi, x)$ are polynomials of degree $2n$ satisfying the conditions $f_{2n}^s(f\xi, tx)=t^{2n}f_{2n}^s(\xi, x)$, $f_{2n}^s(\xi, x)=f_{2n}^s(x, \xi)$.

Formula (1.8) is obtained by expanding the integral terms in the expression (1.6) for $K_s(t, p)$ in Maclaurin series, and the first term of type (1.7) in the series (1.9) of [3]. It can be shown that the series (1.8) converges uniformly in $|\xi|$, $|x|\leq 1$ for $\lambda>\max(1/(2\alpha), 2/\pi)$, i.e. for small α we must have $\alpha>\beta/2$.

The first few terms of the series (1.8) are

$$a_0(s)=-C-\tfrac{1}{2}\psi(\tfrac{1}{2}+s)-\tfrac{1}{2}\psi(\tfrac{1}{2}-s)+\ln 2$$

$$a_1(s)=\tfrac{1}{8}(\tfrac{1}{4}-s^2)[2\psi(2)-\psi(\tfrac{3}{2}+s)-\psi(\tfrac{3}{2}-s)+2\ln 2]+373/9000$$

$$b_0(s)=-1,\quad b_1(s)=-\tfrac{1}{4}(\tfrac{1}{4}-s^2)$$

$$f_0^s(\xi,x)=d_0(s)=\int_0^\infty F(u,s)du\quad (|\operatorname{Re} s|<\tfrac{1}{2})$$

$$F(u,s)=\frac{1}{4\pi^2}\Gamma\left(\frac{1}{4}+\frac{s}{2}+i\frac{u}{2}\right)\Gamma\left(\frac{1}{4}+\frac{s}{2}-i\frac{u}{2}\right)\times$$

$$\times[\operatorname{sh}\pi u(W_m(u)-\operatorname{cth}\pi u)\Gamma\left(\frac{1}{4}-\frac{s}{2}+i\frac{u}{2}\right)\Gamma\left(\frac{1}{4}-\frac{s}{2}-i\frac{u}{2}\right)+$$

$$+2\operatorname{sh}\frac{\pi u}{2}W_m(u)B_m^u\left\{\operatorname{ch}\frac{\pi u}{2}\Gamma\left(\frac{1}{4}-\frac{s}{2}+i\frac{y}{2}\right)\Gamma\left(\frac{1}{4}-\frac{s}{2}-i\frac{y}{2}\right)\right\}\Bigg] \tag{1.9}$$

$$f_2^s(\xi,x)=d_2^1(s)(\xi^2+x^2)+d_2^2(s)\xi x$$

$$d_2^1(s)=\int_0^\infty\left(\frac{1}{4}-s^2+u^2\right)F(u,s)du \quad (|\operatorname{Re}s|<\tfrac{1}{2})$$

$$d_2^2(s)=-\int_0^\infty\frac{\Gamma(\tfrac12+s+iu)(\tfrac12+s+iu)}{\cos\pi[(\tfrac12+s-iu)/2]}P_{\frac12+s}^{-iu}(0)\times$$

$$\times\left[\frac{1}{2}\operatorname{sh}\pi u(W_m(u)-\operatorname{cth}\pi u)\frac{\Gamma(\tfrac12-s+iu)(\tfrac12-s+iu)}{\cos\pi[(\tfrac12-s-iu)/2]}P_{\frac12-s}^{-iu}(0)+\right.$$

$$\left.+\operatorname{sh}\frac{\pi u}{2}W_m(u)B_m^u\left\{\operatorname{ch}\frac{\pi y}{2}\frac{\Gamma(\tfrac12-s+iy)(\tfrac12-s+iy)}{\cos\pi[(\tfrac12-s-iy)/2]}P_{\frac12-s}^{-iy}(0)\right\}\right]du$$

where C is Euler's constant and $\psi(s)$ is the psi-function [12].

An asymptotic solution of the integral equation (1.5) with kernel (1.8) for small β may be obtained by the method of "large λ" [3]. Henceforth, to fix ideas, we shall confine ourselves to the case

$$f_*(\rho,\psi)=f\rho^\mu e^{-\gamma\rho}(\mu\geq\delta-1,\delta>0,\mu>0) \tag{1.10}$$

If $\mu=0$, $\gamma\to0$ the punch degenerates into a flat punch. Using standard integrals [12], we deduce from (1.10) that

$$f_s^*(\psi)=f\gamma^{-(s+\frac12+\mu)}\Gamma(s+\tfrac12+\mu),\quad \operatorname{Re}s>-\tfrac12-\mu \tag{1.11}$$

Suppose that β is so small that one can ignore terms of order λ^{-2} and higher. Then the solution of the problem may be expressed in the form

$$q_*(\rho,\psi)=\frac{f\gamma^{-\mu+1}}{2\pi i\sqrt{\beta^2-\psi^2}}\int_\Gamma\frac{(\rho\gamma)^{-s-\frac32}\Gamma(s+\frac12+\mu)}{g(s)}ds \tag{1.12}$$

$$g(s)=\ln4\lambda-C-\tfrac12\psi(\tfrac12+s)-\tfrac12\psi(\tfrac12-s)+d_0(s) \tag{1.13}$$

From formula (1.12) using the theory of residues, one can obtain an approximate solution of the problem for small β, provided that the zeros of the function $g(s)$ (1.13) are known. The position of the straight line Γ is chosen so as to ensure that condition (1.4) is satisfied and convergence of the integral

$$q_s^*(\psi)=\int_0^\infty q_*(\rho\psi)\rho^{s+\frac12}d\rho \tag{1.14}$$

2. Let us study the zeros of the function $g(s)$ defined by (1.13) in the strip $|\operatorname{Re}s|\leq3/2$, taking into account that, as follows from the lemma, it takes real values on the real and imaginary axes, and $g(-s)=g(s)$. The zeros of the function $g_0(s)=g(s)-d_0(s)$ in the strip have been found [3]. The function $d_0(s)$ defined by (1.9) in $|\operatorname{Re}s|<1/2$ must be considered in the region $|\operatorname{Re}s|\geq1/2$ as an analytic continuation of (1.9). The function $g(s)$ has a simple pole on the real axis at $s=1/2$, with residue $-V_m$ $(m=1,2,3)$, where

$$V_m=\frac{1}{2}+A_m(1+B_m^0\{1\})-\frac{1}{\pi},\ A_m=\lim_{u\to0}uW_m(u) \tag{2.1}$$

Obviously, $V_3 = 1/2 - 1/\pi \approx 0.812$. The following table lists the values of $V_{1,2}$ as functions of $\alpha = \pi k/4$ for $\nu = 0.3$

k	1	2	3	4	5	6	7
V1	8.27	0.969	0.583	0.500	0.493	0.406	0.350
V2	0.649	0.818	0.462	0.182	0.296	0.394	0.283

In addition $V_1 \to +\infty$ and $V_2 \to 1/2 - 1/\pi$ as $\alpha \to 0$.

To compute $V_{1,2}$ by formula (2.1), it is more convenient not to sum the Neumann series $B^0_{1,2}\{1\}$ but to solve the corresponding Fredholm integral equations of the second kind [9] by the method of mechanical quadratures, using Gauss' quadrature formula. This remark also applies to other computations involving the operator B^{μ}_m $(m = 1, 2, 3)$.

It follows from the above calculations that, as a rule, $g(1/2 \pm 0) = \mp\infty$ on the real axis. It can be shown that $g(3/2 - 0) = +\infty$, $g(i\infty) = -\infty$. For sufficiently large values of λ obviously, $g(0) > 0$. Therefore, for a fixed angle α and $\lambda > \lambda_*(\alpha)$, $g(s)$ will have in the strip $|\mathrm{Re}\, s| \leqslant 3/2$ two simple zeros on the imaginary axis and two on the real axis: $s_{1,2} = \pm i\theta$, $\theta = \theta(\alpha, \lambda) = O(\lambda)(\lambda \to \infty)$, $s_{3,4} = \pm(1/2 + \eta)$, $\eta = \eta(\alpha, \lambda) \in (0; 1)$, and moreover $\lambda_*(\alpha) = O(e^{1/(4\alpha)})$ as $\alpha \to 0$. Using the theorem on the zeros of $g_0(s)$ proved in [3] and Rouché's theorem, we conclude that for fixed α, as $\lambda \to \infty$ the zeros of $g(s)$ on the imaginary axis are the only ones in the strip $|\mathrm{Re}\, s| < 1/2$. For example, if $\lambda = 5$, $\nu = 0.3$, $\alpha = \pi k/4$ $(k = 1, 2, \ldots, 7)$ then for all types of boundary conditions (1.1) $(m = 1, 2, 3)$ $s_{1,2} \approx \pm i11.2$, because of the exponential decrease of the function $|\Gamma(z + iy)|$, in expression (1.9) for $d_0(s)$ as $|y| \to \infty$, $x, y \in R$. For small α, comparable with $\beta = 0.2$ (and the same λ, ν), the values of θ are as follows:

α	0.1	0.2	0.4
$m = 1$	19.3	13.0	11.3
$m = 2$	0.0960	10.2	11.2
$m = 3$	–	8.38	11.2

When λ is fixed and $\alpha \to 1/(2\lambda) + 0$, the equation $g(s) = 0$ may have additional complex zeros in the strip $|\mathrm{Re}\, s| < 1/2$. Thus, for an elastic wedge with one rigidly fastened edge $(m = 3)$, at $\alpha = 1.66$, $\lambda = 5$, $\nu = 0.3$, $g(s)$ has eight pure imaginary zeros in the interval $(i2.1, i5.1)$ and three real zeros in (0, 1/2). At $\alpha = 2\beta = 0.4$ ($\lambda = 5$, the elastic wedge angle equals the wedge angle of the punch), $\nu = 0.3$, $m = 3$, there are two additional real roots in the interval (0.46, 0.5).

Let us suppose that the contour Γ in formula (1.12) is contained in the strip $0 < \mathrm{Re}\, s < 1/2$ and intersects the real axis to the right of the zero of $g(s)$ lying in that strip, say $s = \omega_1 + i\omega_2$, with the largest real part ω_1 if there are any such zeros), and also to the right of the point $-(1/2 + \mu)$ if it lies in the interval (0, 1/2). One can then use the theory of residues to find the leading terms of the asymptotic expansion of $q_*(\rho, \psi)$ as $\rho \to 0$. Let us assume from the start that in the strip $|\mathrm{Re}\, s| < 1/2$ the function $g(s)$ has zeros only on the imaginary axis $(\lambda > \lambda_*(\alpha))$. Then, if $\delta - 1 \leqslant \mu < -1/2$, the principal singularity of $q_*(\rho, \psi)$ will be $\rho^{\mu-1}$, second to which come oscillatory singularities $\rho^{-3/2}\cos\theta(\ln\rho\gamma)$ and $\rho^{-3/2}\sin\theta(\ln\rho\gamma)$. If $\mu = -1/2$, the oscillatory singularities prevail.

Thus, in the neighbourhood of the apex of a wedge-shaped punch pressed into the edge of an elastic wedge, the contact conditions may be violated. For an elastic wedge with one stress-free edge $(m = 1)$, the frequency of these oscillations will increase as $\alpha \to 1/(2\lambda) + 0$.

Now suppose that in the strip $0 < \mathrm{Re}\, s < 1/2$ the equation $g(s) = 0$ has an additional complex root $s = \omega_1 + i\omega_2$, $\omega_2 \neq 0$ (α and β are of the same order of magnitude; $g(s) \neq 0$ for $\mathrm{Re}\, s = 1/2$). In that case, if $\delta - 1 \leqslant \mu < -(\omega_1 + 1/2)$, the principal singularity of the contact stress function will be $\rho^{\mu-1}$, second to which will be oscillatory singularities lie that are stronger than previously. If $\mu \geqslant (\omega_1 + 1/2)$, these oscillatory singularities will prevail.

Using Rouché's theorem, one can show that the above qualitative picture remains unchanged if, while using the "large λ" method, one does not ignore terms of order λ^{-2} and higher.

Note that the functions $a_1(s)$, $d_2^1(s)$, $d_2^2(s)$ (see (1.9)), unlike $a_0(s)$ and $d_0(s)$, do not have poles at $s=\pm 1/2$.

Analogous arguments will show that $q_*(\rho, \psi) \sim O(\rho^{-2-\eta})$ as $\rho \to \infty$, where $\operatorname{Re} s = 1/2+\eta$ is the least real part of any root of the equation $g(s)=0$ in the strip $1/2<\operatorname{Re} s<3/2$.

Noting the behaviour of the function $q_*(\rho, \psi)$ as $\rho \to 0$ and $\rho \to \infty$, one sees that the integral (1.4) converges. The integral (1.14) is clearly also convergent, provided that the straight line Γ intersects the real axis slightly to the left of $s=1/2$.

REFERENCES

1. ALEKSANDROV V. M. and BABESHKO V. A., On a punch of wedge-shaped planform pressed into an elastic half-space. *Prikl. Mat. Mekh.* **36**, 1, 88–93, 1972.
2. GALIN L. A., *Contact Problems of the Theory of Elasticity and Viscoelasticity*. Nauka, Moscow, 1980.
3. RVACHEV V. L. and PROTSENKO V. S., *Contact Problems of the Theory of Elasticity for Non-classical Regions*. Naukova Dumka, Kiev, 1989.
4. BABESHKO V. A., GLUSHKOV Ye. V. and ZNICHENKO Zh. F., *Dynamics of Inhomogeneous Linearly Elastic Media*. Nauka, Moscow, 1989.
5. BABESHKO V. A. and PRYAKHINA O. D., A method for solving mixed problems of the theory of elasticity for wedge-shaped regions. *Izv. Severokavok. Nauch. Tsentra Vyosh. Shkol. Yest. Nauki* 1, 21–24, 1982.
6. KEER L. M. and PARIHAR K. S., A note on the singularity at the corner of a wedge-shaped punch or a crack. *SIAM J. Appl. Math.* **34**, 2, 297–302, 1978.
7. PARIHAR K. S. and KEER L. M., The singularity at the apex of a bonded wedge-shaped stamp. *Trans. ASME, Ser. E, J. Appl. Mech.* **46**, 3, 577–580, 1979.
8. BABESHKO V. A., GLUSHKOV Ye. V., GLUSHKOVA N. V. and LAPINA O. N., Singularity of stresses in the neighbourhood of the apex of an elastic trihedron. *Dokl. Akad. Nauk SSSR* **318**, 5, 1113–1116, 1991.
9. LUBYAGIN I. A., POZHARSKII D. A. and CHEBAKOV M. I., Extension of the Boussinesq and Cerruti problems for the case of an elastic three-dimensional wedge. *Dokl. Akad. Nauk SSSR* **321**, 1, 58–61, 1991.
10. LUBYAGIN I. A., POZHARSKII D. A. and CHEBAKOV M. I., Penetration of a punch in the shape of an elliptic paraboloid into an elastic three-dimensional wedge. *Prikl. Mat. Mekh.* **56**, 2, 286–295, 1992.
11. PRUDNIKOV A. P., BRYCHKOV Yu. A. and MARICHEV O. I., *Integrals and Series. Special Functions*. Nauka, Moscow, 1983.
12. GRADSHTEIN I. S. and RYZHIK I. M., *Tables of Integrals, Sums, Series and Products*. Nauka, Moscow, 1971.

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